Water-Filling: A Geometric Approach and its Application to Solve Generalized Radio Resource Allocation Problems

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***Abstract*—In this paper, a simple and elegant geometric water-filling (GWF) approach is proposed to solve the unweighted and weighted radio resource allocation problems. Unlike the conventional water-filling (CWF) algorithm, we eliminate the step to find the water level through solving a non-linear system from the Karush-Kuhn-Tucker conditions of the target problem. The proposed GWF requires less computation than the CWF algorithm, under the same memory requirement and sorted parameters. Furthermore, the proposed GWF avoids complicated derivation, such as derivative or gradient operations in conventional optimization methods, while provides insights to the problems and the exact solutions to the target problems. Most importantly, the GWF can be extended to solve a generalized form of radio resource allocation problem with more stringent constraints: (weighted) optimization problem with individual peak power constraints (GWFPP), and to include (weighted) group bounded power constraints (GWFGBP). On the other side, the CWF cannot solve these two general forms of the RRA problems, due to the difficulty to solve the non-linear system with multiple non-linear equations and inequalities in multiple dual variables. Optimality of the proposed water-filling solution is strictly proved for each of the proposed algorithms. Furthermore, numerical results show that the proposed approach is effective, efficient, easy to follow and insight-seeing.**

***Index Terms*—Water-filling, channel capacity, optimal radio resource allocation, multi-user MIMO (MU-MIMO), cognitive radio, optimization methods.**

I. INTRODUCTION

IN many engineering problems, water-filling plays an important role in radio resource allocation (RRA). For communications, it stems from a class of the problems of maximizing the mutual information between the input and the output of a channel with parallel independent sub-channels. With water-filling, more power is allocated to the channels with higher gains to maximize the sum of data rates or the capacity of all the channels. The solution to this class of the problems can be interpreted by a vivid description as pouring limited volume of water into a tank, the bottom of which has the stair levels determined by the inverse of the sub-channel gains. The principle can be extended to dealing with the issues from communication systems, such as those of a multi-carrier channel [1], [2], a frequency-selective channel [3], a multi-user multiple input multiple output (MIMO) channel, [4]-[7], *etc*. On the one hand, as information of the channel is given, it can obtain the optimal power distribution of the transmitted signal [8], [9]; on the other hand, as information of the transmitted and the corresponding received signals is given, it is equivalent to finding an equalizer [10], [11]. In addition, water-filling has being deepened into the case that only partial knowledge is given [12]; and being extended to solve the problems in the smart grid communication network systems [13]. The focus of this paper is to solve a generic resource allocation problem by utilizing the water-filling principle. For convenience of statement, water-filling (WF) in this paper means two things: a class of optimal resource allocation problems; and the algorithms to compute the exact solution to such a class of the problems.

The conventional water-filling (CWF) problem has a sum power constraint under non-negative individual powers. It can be solved by non-geometric WF approaches. Since the solution is parameterized with a water level, the problem reduces to obtaining the water level such that the power constraint is satisfied with equality. This leads to a non-linear system, in one parameter, such as the water level, that is determined by the sum power constraint and the function $\max(0, x)$. Further, this non-linear system consists of a non-linear equation and another inequality to find the water level. A class of methods to solve the non-linear system are called the conventional WF algorithm. In order to find the exact value of the water level, different algorithms have been proposed that can be classified into iterative algorithms and exact algorithms. The iterative algorithms are trivially implemented in practice and get close to the exact value as the number of iterations goes to infinity [11], [14], [15]. On the other hand, the exact algorithms give the exact solution in a finite number of loops or iterations [7], [16].

For RRA, one of the most typical problems is to solve power allocation using the CWF. When we consider different weight of the channels, the problem can be solved using weighted WF algorithm. As communication system develops, the structures of the system models and the corresponding RRA problems evolve to more advanced and more complicated ones. In this paper, we apply the concept of WF to solve the generalized RRA problems, which include not only the constraints in the CWF but also more stringent constraints, such as i). *W*ater-*F*illing with individual *P*eak *P*ower constraints (WFPP); and

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ii). a kind of more general form, *G*roup *B*ounded *P*ower constraints (WFGBP). For the WFPP, we include upper power constraints for each channel. For the WFGBP, the channels are categories into groups. We enforce the constraints that the sum power for each group is lower and upper bounded. The extended RRA problem can find its applications in the advanced communication systems. For instance, for the WFPP, in the uplink, the fact that the transmit power of the mobile user is limited leads to individual peak power constraints. Typically, in a cognitive radio (CR) network, the objective of the design is to maximize the sum rate of the second users (SUs) which have individual peak power constraints. At the same time, in order to guarantee the quality of service (QoS) of the primary users (PUs), the interference from the SUs to the PUs is forced to be below a threshold [17], [18]. For the WFGBP, the fact that the transmit powers of the groups of the mobile users are bounded leads to group bounded power constraints [19]-[20]. Another application for the WFGBP problem is the requirement of the (proportional) fair rate among all the groups from the SUs. This problem can be transformed to a set of group lower bounded power constraints.

In this paper, as the first step, a geometric water-filling (**GWF**) approach is proposed to solve the conventional water-filling problem and its weighted form. It has two advantages. On the one side, the geometric approach can compute the exact solution to the CWF, including the weighted case, with less computation and easier analysis without determining the water level through solving the non-linear system. On the other side, machinery of the proposed geometric approach can overcome the limitations of the CWF algorithm to include more stringent constraints. Thus, as the second step, applying the concept of water-filling and the proposed geometric machinery, we extend the proposed **GWF** to solve more generalized WFPP and WFGBP problems. In numerical results section, it is shown that for the generalized WFPP and WFGBP problems, with optimal power allocation, the water levels for different channels can be different. Thus, the CWF method cannot solve this kind of generalized problems through determining a unified water level.

For the WFPP problem, when this paper was completed, we found two research papers [21], [22] on water-filling published recently where the individual peak power constraints were applied. The former follows the typical water level searching approach and the latter removes the traditional water level searching. They belong to the non-geometric approach. In contrast to these works, a novel geometric WF approach to directly compute the solution to a family of general RRA problems is independently proposed in this paper. The proposed algorithms are more general with simple procedures and solutions. Interesting insights of the relationship among the physical quantities can be revealed. The optimality of the proposed water-filling solution is provided.

In the remaining of the paper, the problem statement, the CWF and the proposed **GWF** are discussed in Section II with sum power constraint, including unweighted problem and weighted problem. The generalized weighted water-filling problems with additional more stringent constraints, *i.e.*, the WFPP and the WFGBP, are further investigated in Section III. Numerical examples are presented and the solutions are illustrated for the CWF and the proposed **GWFPP** and **GWFGBP** in Section IV, followed by complexity analysis of the discussed algorithms. Section V concludes the paper. Appendix gives a list of used variables and abbreviations.

II. Water-Filling with Sum Power Constraint

A. Problem Statement and Conventional Approach

The water-filling problem can be abstracted and generalized into the following problem: given $P > 0$, as the total power or volume of the water; the allocated power and the propagation path gain for the ith channel are given as s_i and a_i respectively, $i = 1 \ldots K$; and K is the total number of channels. Let $\{a_i\}_{i=1}^K$ be a sorted sequence, which is positive and monotonically decreasing, find that

$$\begin{array}{ll} \max_{\{s_i\}_{i=1}^K} & \sum_{i=1}^K \log(1+a_i s_i) \\ \text{subject to:} & 0 \le s_i, \ \forall i; \\ & \sum_{i=1}^K s_i = P. \end{array} \tag{1}$$

Since the constraints are that (i) the allocated power to be nonnegative; (ii) the sum of the power equals P, the problem (1) is called the water-filling (problem) with sum power constraint.

To find the solution to problem (1), we usually start from the Karush-Kuhn-Tucker (KKT) conditions of the problem, as a group of the optimality conditions, and derive the system (2) below from the KKT conditions,

$$\begin{cases} s_i = \left(\mu - \frac{1}{a_i}\right)^+, \text{ for } i = 1, \ldots, K, \\ \sum_{i=1}^K s_i = P, \\ \mu \ge 0, \end{cases} \tag{2}$$

where $(x)^+ = \max\{0, x\}$. μ is the water level chosen to satisfy the power sum constraints with equality ($\sum_{i=1}^K s_i = P$). The solution to (2) is referred as a solution of the CWF problem (1).

It can be seen that the implied system (2) has been used to find the optimal solution. The existence of its Lagrange multipliers and the implication mentioned above determine that enumeration can be utilized to find the water level μ. In [16], how to solve the problems has been discussed extensively. Complexity of the non-geometric approach to solve the problem (1) will be discussed in Section IV. In the sequel of the paper, when water-filling problem is mentioned, the power sum constraint is always included.

B. Proposed Geometric Water-Filling (GWF) Approach

In this paper, we propose a novel approach to solve problem (1) based on geometric view. The proposed Geometric Water-Filling (**GWF**) approach eliminates the procedure to solve the non-linear system for the water level, and provides explicit solutions and helpful insights to the problem and the solution.

Figs. 1(a)-(c) give an illustration of the proposed **GWF** algorithm. Suppose there are 4 steps/stairs ($K = 4$) with unit width inside a water tank. For the conventional approach, the dashed horizontal line, which is the water level μ, needs to be determined first and then the power allocated for each stair (water volume above the stair) is solved.

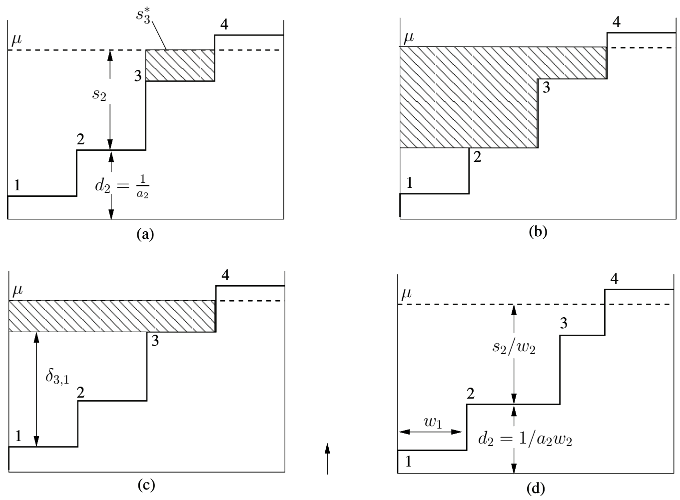

Fig. 1. Illustration for the proposed Geometric Water-Filling (**GWF**) algorithm. (a) Illustration of water level step $k^* = 3$, allocated power for the third step s_3^*, and step/stair depth $d_i = 1/a_i$. (b) Illustration of $P_2(k)$ (shadowed area, representing the total water/power above step k) when $k = 2$. (c) Illustration of $P_2(k)$ when $k = 3$. (d) Illustration of the weighted case.

Let us use d_i to denote the "step depth" of the ith stair which is the height of the ith step to the bottom of the tank, and is given as

$$d_i = \frac{1}{a_i}, \text{ for } i = 1, 2, \ldots, K. \tag{3}$$

Since the sequence a_i is sorted as monotonically decreasing, the step depth of the stairs indexed as $\{1, \cdots, K\}$ is monotonically increasing. We further define $\delta_{i,j}$ as the "step depth difference" of the ith and the jth stairs, expressed as

$$\delta_{i,j} = d_i - d_j = 1/a_i - 1/a_j, \text{ as } i \geq j \text{ and } 1 \leq i, j \leq K. \tag{4}$$

Instead of trying to determine the water level μ, which is a real nonnegative number, we aim to determine water level step, which is an integer number from 1 to K, denoted by k^*, as the highest step under water. Based on the result of k^*, we can write out the solutions for power allocation instantly.

Fig. 1(a) illustrates the concept of k^*. Since the third level is the highest level under water, we have $k^* = 3$. The shaded area denotes the allocated power for the third step by s_3^*.

In the following, we explain how to find the water level step k^* without the knowledge of the water level μ. Let $P_2(k)$ denote the water volume above step k or zero, whichever is greater. The value of $P_2(k)$ can be solved by subtracting the volume of the water under step k from the total power P, as

$$\begin{aligned} P_2(k) = & \left\{P - \left[\sum_{i=1}^{k-1}\left(\frac{1}{a_k} - \frac{1}{a_i}\right)\right]\right\}^+ \\ = & \left\{P - \left[\sum_{i=1}^{k-1}\delta_{k,i}\right]\right\}^+, \text{ for } k = 1, \ldots, K. \end{aligned} \tag{5}$$

Due to the definition of $P_2(k)$ being the power (water volume) above step k, it cann't be a negative number. Therefore we use $\{\cdot\}^+$ in (5) to assign 0 to $P_2(k)$ if the result inside the bracket is negative. The corresponding geometric meaning is that the kth level is above water. Note a reminder of the definition of a special case for the summation is:

$$\sum_{i=m}^{n} b_i = 0, \text{ as } m > n. \tag{6}$$

Fig. 1(b) and Fig. 1(c) illustrate the concept of $P_2(k)$ for $k = 2$ and $k = 3$ respectively by the shadowed area. As an example of Fig.1(c), the water volume under step 3 can be expressed as the sum of the two terms: (i) the step depth difference between the 3rd and the 1st step, $\delta_{3,1}$, and (ii) the step depth difference between the 3rd and the 2nd step, $\delta_{3,2}$. Thus, $P_2(k = 3)$ can be written as

$$P_2(k = 3) = [P - \delta_{3,1} - \delta_{3,2}]^+$$

and the above result is the shadowed area in Fig. 1(c), which is also an expansion of the composite form of (5). Then, we are ready to have the following proposition:

Proposition 2.1. The explicit solution to (1) is:

$$s_i = \begin{cases} s_{k^*} + (d_{k^*} - d_i) & 1 \leq i \leq k^* \\ 0, & k^* < i \leq K, \end{cases} \tag{7}$$

where the water level step k^* is given as

$$k^* = \max\left\{k \middle| P_2(k) > 0, \;\; 1 \leq k \leq K\right\} \tag{8}$$

and the power level for this step is

$$s_{k^*} = \frac{1}{k^*} P_2(k^*). \tag{9}$$

It is easy to interpret **Proposition 2.1** from Fig. 1. The first step of the proposed approach is to find the water level step k^*. From Fig. 1, we can find that $k = 3$ is the maximal index that makes $P_2(k)$ greater than zero. Therefore, based on (8), $k^* = 3$ can be determined. Then the power at this step s_{k^*} can be determined based on (9). For those steps with index higher than k^*, no power is assigned. For those steps with index lower than k^*, their power levels are obtained by adding s_{k^*} with the corresponding level depth difference with the k^*th step as shown in (7).

Proposition 2.1 provides an explicit constructed solution rather than the implicit solution. The procedure eliminates solving the nonlinear equation as shown in (2) and the real number water level μ. The proof of the optimality of the solution will be left to the next subsection when we discuss the weighted case.

C. Generalize to Weighted Case

For the weighted case, the generalized problem can be stated as: given $P > 0$, as the total power or volume of the water; the allocated power and the propagation path gain for the ith antenna are given as s_i and a_i respectively, $i = 1, \ldots, K$; and K is the total number of the transmit antenna. Furthermore, the weighted coefficients $w_i > 0, i \in \{1, \ldots, K\}$, and $\{a_i w_i\}_{i=1}^K$ being monotonically decreasing, find that

$$\begin{array}{ll} \max_{\{s_i\}_{i=1}^K} & \sum_{i=1}^K w_i \log(1 + a_i s_i) \\ \text{subject to:} & 0 \leq s_i, \;\forall i; \\ & \sum_{i=1}^K s_i = P. \end{array} \tag{10}$$

Using the proposed geometric approach, we can extend the geometric relation for the weighted case as shown in Fig.1(d) to obtain the corresponding solution to (10).

In Fig. 1(d), the width of the ith stair/step is denoted as w_i. The value of $1/a_i$ denotes the volume under the ith step to the bottom of the tank. Hence, the step depth of the ith step is given as

$$d_i = 1/(a_i w_i), \quad i = 1, \cdots, K. \tag{11}$$

Then, $P_2(k)$, the water volume above step k, can be obtained using the similar approach as in the previous subsection considering the step depth difference and the width of the stairs as,

$$P_2(k) = \left[P - \sum_{i=1}^{k-1} (d_k - d_i) w_i\right]^+, \text{ for } k = 1, \ldots, K. \tag{12}$$

As an example in Fig. 1(d), the water volume above step 1 and below step 3 with the width w_1 can be found as: the step depth difference, $(d_3 - d_1)$ multiplying the width of the step, w_1. Therefore, the corresponding $P_2(k = 3)$ can be expressed as,

$$P_2(k = 3) = [P - (d_3 - d_1)w_1 - (d_3 - d_2)w_2]^+,$$

which is an expansion of (12). Then we have the following proposition.

Proposition 2.2. The explicit solution to (10) is:

$$\begin{cases} s_i = \left[\dfrac{s_{k^*}}{w_{k^*}} + (d_{k^*} - d_i)\right] w_i, & 1 \le i \le k^* \\ s_i = 0, & k^* < i \le K, \end{cases} \tag{13}$$

where

$$k^* = \max\left\{k \middle| P_2(k) > 0, \ 1 \le k \le K\right\} \tag{14}$$

and the power level for this step is

$$s_{k^*} = \frac{w_{k^*}}{\sum_{i=1}^{k^*} w_i} P_2(k^*). \tag{15}$$

Proof of Proposition 2.2. System (13) implies that

$$\frac{w_{k^*}}{\frac{1}{a_{k^*}} + s_{k^*}} = \frac{w_i}{\frac{1}{a_i} + s_i}, \text{ as } 1 \le i \le k^*. \tag{16}$$

Let

$$\lambda = \frac{w_{k^*}}{\frac{1}{a_{k^*}} + s_{k^*}}. \tag{17}$$

From a geometric view, λ is the reciprocal of water level μ. According to the definitions of k^* and s_{k^*}, for $k^* < i \le K$, $\frac{w_{k^*}}{\frac{1}{a_{k^*}} + s_{k^*}} > \frac{w_i}{\frac{1}{a_i} + s_i}$ and $s_i = 0$.

Let $\sigma_i = \frac{w_{k^*}}{\frac{1}{a_{k^*}} + s_{k^*}} - \frac{w_i}{\frac{1}{a_i} + s_i}$. Then

$$\begin{cases} \sigma_i > 0, \text{ as } k^* < i \le K \\ \sigma_i = 0, \text{ as } 1 \le i \le k^*. \end{cases} \tag{18}$$

Therefore, the following system holds:

$$\begin{cases} \frac{w_i}{\frac{1}{a_i} + s_i} - \lambda + \sigma_i = 0, & \text{as } 1 \le i \le K \\ s_i \ge 0, & \forall i \\ \sigma_i s_i = 0, & \forall i \\ \sigma_i \ge 0, & \forall i \\ \sum_{i=1}^{K} s_i = P, & \lambda \in \mathbb{R}. \end{cases} \tag{19}$$

By observation, the equation and inequality set above is just a set of the KKT conditions of the problem in **Proposition 2.2** and the water level μ is equal to the reciprocal of the Lagrange multiplier λ mentioned above. Note that the Lagrange function of the problem in **Proposition 2.2** is

$$L(\{s_i\}, \lambda, \{\sigma_i\}) = \sum_{i=1}^{K} w_i \log(1 + a_i s_i) - \lambda \left(\sum_{i=1}^{K} s_i - P\right) + \sum_{i=1}^{K} \sigma_i s_i. \tag{20}$$

Since it is a differentiable convex optimization problem with linear constraints, not only are the **KKT** conditions mentioned above sufficient, but they are also necessary for optimality. Note that the constraint qualification of the problem (10) holds. **Proposition 2.2** hence is proved.

Similar to the unweighted case, the first step is to calculate $P_2(k)$, then find the water level step, k^*, from (14), which is the maximal index making $P_2(k)$ nonnegative. The corresponding power level for this step, s_{k^*}, can be obtained by applying (15). Then for those steps with index higher than k^*, the power level is assigned with zero. For those steps below k^*, the power level is assigned as in (13). The first term (s_{k^*}/w_{k^*}) inside the square bracket denotes the depth of the k^*th step to the surface of the water. The second term inside the square bracket denotes the step depth difference of the k^*th step and the ith step. Therefore, the sum inside the square bracket means the depth of the ith step to the surface of the water. When this quantity is multiplied with the width of this step, the volume of the water above this step (allocated power) can be then readily obtained.

With the proposed **GWF** approach, the weighted problem could be solved straightforwardly, avoiding complicated derivation and calculation. When the weighting factors are set to ones, the corresponding unweighted case is obtained. In the following description of algorithm implementation and proof, we only provide weighted case.

From **Proposition 2.2**, when k^* is obtained, $P_2(k^*)$ is given. Then it is memorized and only multiplied by a constant to compute s_{k^*}. Thus, how to search k^* is a key point for the proposed **GWF** and the procedure is stated as follows:

1) Initialize $W_s = 0; P_M = P^* = P; i = 1$.
2) Compute $W_s <= W_s + w_i; P^* <= P^* - (d_{i+1} - d_i)W_s$. Then $i <= i + 1$, where the symbol "<=" represents the assignment operation.
3) If $P^* > 0$ and $i \le K$, $P_M = P^*$, and repeat the step 2); else, output $k^* = i - 1, W_s = W_s - w_i$ and $s_{k^*} = \frac{w_{k^*}}{W_s} P_M$.

We can observe that $\frac{s_{k^*}}{w_{k^*}} + d_{k^*}$ is the water level due to $\frac{s_{k^*}}{w_{k^*}} + d_{k^*} = \frac{s_i}{w_i} + d_i$, for $1 \le i \le k^*$.

As an alternative to the enumeration search in the Algorithm **GWF**, a Fibonacci-like search is possibly used to speed up finding k^* due to (non-increasing) monotonicity of the sequence $\{P_2(k)\}$. Without loss of generality, let Fibonacci approximation ratios be $\frac{1}{3}$ and $\frac{2}{3}$ for searching k^*. The method can be described as:

1st Step. Assume that $a = 1$ and $b = K$.

2nd Step. If $a = b$, then $k^* = a$ and go to **Step 3** of GWF. Else, $a_1 = \lfloor a + \frac{1}{3}(b - a) \rfloor$, $b_1 = \lceil a + \frac{2}{3}(b - a) \rceil$.

3rd Step. If $P_2(a_1) \le 0$, then $b = a_1 - 1$ and go to the **2nd Step**;
If $P_2(b_1) > 0$, then $a = b_1$ and go to the **2nd Step**;

If $P_2(a_1) > 0$ and $P_2(b_1) \leq 0$, then $a = a_1$, $b = b_1 - 1$ and go to the **2nd Step**.

The number of loops to search k^* is reduced into a complexity level of $\log_3(K)$.

III. Solving Generalized RRA Problems Using GWF Approach

In this section, we firstly extend the CWF problem to include individual peak power constraints (WFPP). Then, we extend the problem to a more generalized form, water-filling with group bounded power constraints (WFGBP). In the WFGBP, if the groups are regressed into singletons, the case with the sum and bounded group power constraints is also regressed into the case with the sum and bounded individual power constraints. Furthermore, if the group bounded power constraints take zero as the lower bounds, they become the individual peak power constraints, *ie.*, those in the form of the WFPP. Therefore, the case with group bounded power constraints is a more general RRA problem. As far as the authors' knowledge, this generalized system model and the algorithm to solve it haven't been reported in the open literatures.

A. Weighted Water-Filling with Individual Peak Power Constraints (WFPP)

The weighted WFPP problem is stated as follows. Given $P > 0$, as the total power or volume of the water; the allocated power and the propagation path gain for the ith antenna are given as s_i and a_i respectively, $i = 1, \ldots, K$; and K is the total number of the transmit antenna. Also, the weights $w_i > 0, \forall i$, and without loss of generality, $\{a_i \cdot w_i\}_{i=1}^K$ being positive and monotonically decreasing, find that

$$\begin{array}{ll} \max_{\{s_i\}_{i=1}^K} & \sum_{i=1}^K w_i \log(1 + a_i s_i) \\ \text{subject to:} & 0 \leq s_i \leq P_i,\ \forall i; \\ & \sum_{i=1}^K s_i \leq P. \end{array} \tag{21}$$

Compare the problem (21) with (10), the constraint of $0 \leq s_i$ is extended to $0 \leq s_i \leq P_i$, *i.e.*, additional individual peak power constraints, and $\sum_{i=1}^K s_i = P$ to $\sum_{i=1}^K s_i \leq P$. The problem (21) is thus referred to as (weighted) water-filling with sum and individual peak power constraints (WFPP). In this section, we discuss the solution to the WFPP problem.

Proposition 2.2 in subsection II-C provides an explicit solution using geometric view approach. Interestingly, the proposed **GWF** can be applied to the WFPP problem with some modifications. The following presents an algorithm which is a modification of the above discussed **GWF** and it is termed as the **GWFPP**.

For convenience, the expression (12) can be extended into the expression:

$$P_2(i_k) = \left[P - \sum_{t=1}^{|E|-1} \left(\frac{1}{d_{i_k}} - \frac{1}{d_{i_t}}\right) w_{i_t}\right]^+, \quad \text{for } k = 1, \ldots, |E|,$$

where E is a subsequence of the sequence $\{1, 2, \ldots, K\}$, $|E|$ is the cardinality of the set E, so E can be expressed as $\{i_1, i_2, \ldots, i_{|E|}\}$. Especially, if E is taken as the sequence $\{1, 2, \ldots, K\}$, then the extended expression is regressed into the original expression (12). Similarly, some corresponding changes in (13)-(15) are also done (*i.e.*, the subscripts of sequence are replaced with those of the subsequence). For avoiding tediousness, these extended expressions are still labelled as (13)-(15) in the following statement of Algorithm **GWFPP**.

Algorithm GWFPP:

Input: vector $\{d_i\}, \{w_i\}, \{P_i\}$ for $i = 1, 2, \ldots, K$, the set $E = \{1, 2, \ldots, K\}$, and P.

1) utilize (13)-(15) compute $\{s_i\}$.
2) The set Λ is defined by the set $\{i | s_i > P_i, i \in E\}$. If Λ is the empty set, output $\{s_i\}_{i=1}^K$; else, $s_i = P_i$, as $i \in \Lambda$.
3) Update E with $E \setminus \Lambda$ and P with $P - \sum_{t \in \Lambda} P_t$. Then return to 1) of the **GWFPP**.

Remark 3.1. Algorithm **GWFPP** is a dynamic power distribution process. The state of this process is the difference between the individual peak power sequence and the current power distribution sequence obtained by the Algorithm **GWF**. The control of this process is to use (13)-(15) of the Algorithm **GWF** based on the state mentioned above. Thus, a new state for next time stage appears. Therefore, an optimal dynamic power distribution process, the **GWFPP**, with the state feedback is formed. Since the finite set E is getting smaller and smaller until the set Λ is empty, Algorithm **GWFPP** carries out K loops to compute the optimal solution, at most.

Similar to the proof of the **Proposition 2.2**, we can obtain the following conclusion:

Proposition 3.1: Algorithm **GWFPP** can provide the optimal solution to the problem (21).

Proof of Proposition 3.1. If the final set E in Algorithm **GWFPP** is empty, it implies that $\sum_{i=1}^K P_i \leq P$. Then it is easy to see the optimal solution $s_i = P_i$, for any i.

If it is non-empty, it implies that

$$\frac{w_{k^*}}{\frac{1}{a_{k^*}} + s_{k^*}} = \frac{w_i}{\frac{1}{a_i} + s_i}, \text{ as } \{i, k^*\} \subset E \text{ and } 0 < s_i \leq P_i. \tag{22}$$

Let

$$\lambda = \frac{w_{k^*}}{\frac{1}{a_{k^*}} + s_{k^*}}, \tag{23}$$

it is seen that

$$\underline{\sigma}_i = \frac{w_{k^*}}{\frac{1}{a_{k^*}} + s_{k^*}} - \frac{w_i}{\frac{1}{a_i} + s_i} \geq 0, \tag{24}$$

and let $\overline{\sigma}_i = 0$, as $i \in E$.

If $i \notin E$, then $s_i = P_i$. According to the definitions of k^* and s_{k^*}, we have the following relationship:

$$\frac{w_{k^*}}{\frac{1}{a_{k^*}} + s_{k^*}} < \frac{w_i}{\frac{1}{a_i} + s_i}, \text{ as } i \notin E, \textit{i.e.}, s_i = P_i. \tag{25}$$

It is seen that

$$\overline{\sigma}_i = \frac{w_i}{\frac{1}{a_i} + s_i} - \frac{w_{k^*}}{\frac{1}{a_{k^*}} + s_{k^*}} > 0 \tag{26}$$

and let $\underline{\sigma}_i = 0$, as $i \notin E$. Then,

$$\begin{cases} \underline{\sigma}_i = 0 \text{ and } \overline{\sigma}_i > 0, \text{ as } i \notin E \\ \underline{\sigma}_i > 0 \text{ and } \overline{\sigma}_i = 0, \text{ as } i \in E \text{ and } s_i = 0 \\ \underline{\sigma}_i = \overline{\sigma}_i = 0, \text{ as } i \in E \text{ and } 0 < s_i \leq P_i. \end{cases} \tag{27}$$

Therefore, the following system holds:

$$\left\{\begin{array}{ll}\frac{w_i}{\frac{1}{a_i}+s_i}-\lambda-\overline{\sigma}_i+\underline{\sigma}_i=0, & \text{as } 1\leq i\leq K\\ s_i\geq 0, \underline{\sigma}_i s_i=0,\ \underline{\sigma}_i\geq 0, & \forall i\\ s_i\leq P_i, \overline{\sigma}_i(s_i-P_i)=0,\ \overline{\sigma}_i\geq 0, & \forall i\\ \sum_{i=1}^K s_i=P, & \lambda\in\mathbb{R}.\end{array}\right. \tag{28}$$

By observation, the system above is just a set of the KKT conditions of the problem in **Proposition 3.1** and the water level μ is equal to the reciprocal of the Lagrange multiplier λ mentioned above. Note that the Lagrange function of the problem in **Proposition 3.1** is

$$\begin{aligned}&L(\{s_i\},\lambda,\{\overline{\sigma}_i\},\{\underline{\sigma}_i\})\\ =\ &\sum_{i=1}^K w_i\log(1+a_i s_i)-\lambda\left(\sum_{i=1}^K s_i-P\right)\\ -\ &\sum_{i=1}^K\overline{\sigma}_i(s_i-P_i)+\sum_{i=1}^K\underline{\sigma}_i s_i.\end{aligned} \tag{29}$$

Since it is a differentiable convex optimization problem with linear constraints, not only are the KKT conditions mentioned above sufficient, but they are also necessary for optimality. Note that the constraint qualification of the problem (21) holds. **Proposition 3.1** is hence proved.

B. Weighted Water-Filling with Group Bounded Power Constraints (WFGBP)

The weighted WFGBP problem is stated as follows. Considering a cognitive network, given $P\geq 0$, as the total power of the CRs or volume of the water; the allocated power, the propagation path gain and the weight for the ith CR are given as s_k, a_k and $w_k(\geq 0)$ respectively, $k=1,\ldots,K$, where K is the total number of the CRs; and let $\{\chi_i\}_{i=1}^T$ be a partition of the index set: $\{1,\ldots,K\}$. For convenience, the elements of χ_i can be listed, monotonically increasing, *i.e.*, $i_1<i_2<\ldots<i_{|\chi_i|}$. $\underline{P}_i$ and $\overline{P}_i$, under the assumption of $0\leq\underline{P}_i\leq\overline{P}_i$, denote the lower bound and the upper bound of the power constraints for the ith group of the CRs, $\forall i$. The generalized weighted water-filling problem with group bounded power constraints under consideration then reads

$$\begin{array}{ll}\max_{\{s_k\}_{k=1}^K} & \sum_{k=1}^K w_k\log(1+a_k s_k)\\ \text{subject to:} & 0\leq s_k,\ \forall k;\\ & \sum_{k=1}^K s_k\leq P;\\ & \underline{P}_i\leq\sum_{k\in\chi_i}s_k\leq\overline{P}_i, i=1,\ldots,T.\end{array} \tag{30}$$

Compared the problem (30) with (21), the constraints of $0\leq s_i\leq P_i,\forall i$, are generalized to $\underline{P}_i\leq\sum_{k\in\chi_i}s_k\leq\overline{P}_k$, *i.e.*, additional group bounded power constraints. The lower bounds of the additional constraints can be used to guarantee the fair transmitted rate from the ith group of CRs, whereas the upper bounds of the additional constraints can be used to limit interference of the group with the primary users, for any i. The problem (30) is thus referred to as (weighted) water-filling with group bounded power constraints (WFGBP). In this subsection, we discuss the solution to the WFGBP problem.

Due to the explicit solution using geometric view approach that is provided in **Proposition 2.2**, interestingly, the proposed **GWF** can be applied to the WFGBP problem with some modifications. The following presents an extended algorithm, which is a meaningful modification of the **GWF** and is termed as the **GWFGBP**.

Note, as $P\leq\sum_{i=1}^T\underline{P}_i$, it is easy to see that there does not exist any solution to problem (30); whereas, as $\sum_{i=1}^T\overline{P}_i\leq P$, problem (30) is regressed into a trivial case without the sum power constraint. Hence, $\sum_{i=1}^T\underline{P}_i\leq P\leq\sum_{i=1}^T\overline{P}_i$ is assumed.

If $\underline{P}_i=0,\overline{P}_i>>0,\forall i$, and the weights are equal, then problem (30) is reduced into the regular case that can be solved by the conventional weighted water-filling problem [7]; and if χ_i is regressed to a singleton and $\underline{P}_i=0,\forall i$, then problem (30) is reduced into the WFPP problem. Thus, (30) is a more general form of the RRA problem.

To find the solution to (30), the generalized geometric water-filling algorithm for the group bounded power constraints (**GWFGBP**) is presented as follows: Firstly, for integrity of this new algorithm, let us re-visit the four concepts: (i) power (water volume) above step k, $P_2(k)$; (ii) power allocated to the ith group tth channel, s_{i_t}; (iii) water level step k^*; and (vi) power allocated for the water level step $s_{i_{k^*}}$ as below:

$$\begin{array}{l}P_2(k)=\left[P-\sum_{t=1}^{|E|-1}\left(\frac{1}{a_{i_k}w_{i_k}}-\frac{1}{a_{i_t}w_{i_t}}\right)w_{i_t}\right]^+,\\ \text{for } k=1,\ldots,|E|,\end{array} \tag{31}$$

where E is a subsequence of the sequence $\{1,2,\ldots,K\}$, $|E|$ is the cardinality of the set E, so E can be written as $\{i_1,i_2,\ldots,i_{|E|}\}$. Note that k in $P_2(k)$ is a subscript of the subsequence $\{i_t\}_{t=1}^{|E|}$ under the assumption: $1\leq i_1<i_2<\ldots<i_{|E|}\leq K$ in the given set E, and the sequence $\{1,2,\ldots,K\}$ is a subsequence of itself under the definition of subsequence.

Also, note

$$s_{i_t}=\left\{\begin{array}{l}w_{i_t}\left(\left(\frac{s_{i_{k^*}}}{w_{i_{k^*}}}+\frac{1}{a_{i_{k^*}}w_{i_{k^*}}}\right)-\frac{1}{a_{i_t}w_{i_t}}\right),\ \text{as } 1\leq t\leq k^*\\ 0,\ \text{as } k^*<t\leq|E|,\end{array}\right. \tag{32}$$

where the water level step k^* is given as

$$k^*=\max\left\{k\middle|P_2(k)>0,\ \ 1\leq k\leq|E|\right\} \tag{33}$$

and the power level for this step is

$$s_{i_{k^*}}=\frac{w_{i_{k^*}}}{\sum_{t=1}^{k^*}w_{i_t}}P_2(k^*). \tag{34}$$

If water-filling is vividly described as pouring the water of volume P into a tank with the bottom of $|E|$ stairs, then $P_2(k)$ is the water volume above the kth stair. Using these four concepts, the steps of the **GWFGBP** can be described as below.

Algorithm GWFGBP:

Input: the channel gains $\{a_k\}_{k=1}^K$, the weights $\{w_k\}_{k=1}^K$, the group lower and upper power bounds $\{\underline{P}_i,\overline{P}_i\}_{i=1}^T$, the index set $E=(E_0=)\{1,2,\ldots,K\}$, the partition $\{\chi_i\}_{i=1}^T$, the sum power constraint P and $i=1$.

1) Initialize $W_{i_s}=0;P_M=P^*=\underline{P}_i;j=1$.
2) Update W_{i_s} with $W_{i_s}+w_{i_j}$ and P^* with $P^*-(d_{i_{j+1}}-d_{i_j})W_{i_s}$. Then increase the iteration index j to $j+1$, where the used symbols are referred to in **Proposition 2.2.**

3) If $P^* > 0$ and $j \le |\chi_i|$, $P_M = P^*$, and repeat the step 2); else, output $k^* = j-1, W_s = W_s - w_{i_j}$, $s_{i_{k^*}} = \frac{w_{i_{k^*}}}{W_{i_s}} P_M$, increase the iteration index i to $i+1$, and then repeat the above process from the step 1), until $i = T$. Thus, $\{s_k\}_{k=1}^K$ is obtained. Let E be updated with $\{1, \ldots, K\}$, P_t with P and $\frac{1}{a_k}$ with $\frac{1}{a_k} + s_k, \forall k$. Finally in this step, let $n = 1$ and $\Lambda = \varnothing$, where $\varnothing$ stands for the empty set.
4) Then utilize (31)-(34) to compute $\{s_i\}$, which appear in the left hand-side (LHS) of (32). Successively, assign $\triangle s_k$ with $s_k, \forall k$.
5) The set Λ_n is defined by the set $\{i | \sum_{j\in\chi_i} \triangle s_j > \overline{P}_i - \underline{P}_i, 1 \le i \le T\}$. If Λ_n is the empty set, output the solution $\{s_k + \triangle s_k\}_{k=1}^K$ to the problem (30); else, let $\sum_{j\in\chi_i} \triangle s_j = \overline{P}_i - \underline{P}_i$, as $i \in \Lambda_n$. Further, continuously utilize similar expressions to (31)-(34), these similar expressions only changing from P_t to $\overline{P}_i - \underline{P}_i$ and from E to χ_i for any $i \in \Lambda_n$, and then obtain $\triangle s_j, j \in \cup_{i\in\Lambda_n} \chi_i$. Let $\Lambda = \Lambda \cup \Lambda_n$.
6) Update E with $E \backslash (\cup_{i\in\Lambda_n} \chi_i)$; P_t with $P_t - \sum_{i\in\Lambda_n} (\overline{P}_i - \underline{P}_i)$. Then increase the iteration index n to $n+1$, and return to 4) of the **GWFGBP**.

Remark 3.2. Algorithm **GWFGBP** is also a dynamic power distribution process. The state of this process is the difference between the group bounded power sequence and current power distribution sequence obtained by (31)-(34). The control of this process is to use the mentioned similar expressions to (31)-(34) based on the state mentioned above. Thus, a new state for next time stage appears. Therefore, an optimal dynamic power distribution process, the **GWFGBP**, with the state feedback is formed. Since the finite set E is getting smaller and smaller until there exists n such that the set Λ_n is empty, Algorithm **GWFGBP** carries out T loops to compute the optimal solution, at most.

For optimality of the proposed algorithm **GWFGBP**, we can obtain the following conclusion:

Proposition 3.2: Algorithm **GWFGBP** can provide the exact optimal solution to the problem (30) via finite computation.

Proof of Proposition 3.2. Without loss of generality, assume that the final set Λ in Algorithm **GWFGBP** is empty. It is easy to see the optimal solutions $\{\triangle s_j\}_{j\in\chi_i}$, for any i, which only require to satisfy $\sum_{j\in\chi_i} \triangle s_j \le \overline{P}_i - \underline{P}_i$ and also satisfy the total sum power constraint for $\{\triangle s_k\}$. Thus, appending all the groups of the solutions from the **GWFGBP**, we can obtain the solution to the problem (30) and its optimality is proven as follows.

The final set E, as a non-empty set, implies that

$$\frac{1}{\frac{1}{a_{k^*} w_{k^*}} + \frac{s_{k^*} + \triangle s_{k^*}}{w_{k^*}}} = \frac{1}{\frac{1}{a_j w_j} + \frac{s_j + \triangle s_j}{w_j}}, \text{ as } \{j, k^*\} \subset E, \tag{35}$$

and there exists χ_i such that $\sum_{j\in\chi_i} \triangle s_j > 0$. Thus, under $\sum_{j\in\chi_i} \triangle s_j > 0$, let

$$\lambda = \frac{1}{\frac{1}{a_{k^*} w_{k^*}} + \frac{s_{k^*} + \triangle s_{k^*}}{w_{k^*}}} \tag{36}$$

and then $\mu_j = 0$ as $\triangle s_j > 0$ and $j \in \chi_i$. Further, according to the definitions of k^* and s_{k^*}, for $j \in E$ and $\triangle s_j = 0$, since

$$\frac{1}{\frac{1}{a_{k^*} w_{k^*}} + \frac{s_{k^*} + \triangle s_{k^*}}{w_{k^*}}} > \frac{1}{\frac{1}{a_j w_j} + \frac{s_j + \triangle s_j}{w_j}}, \tag{37}$$

then let

$$\mu_j = \frac{1}{\frac{1}{a_{k^*} w_{k^*}} + \frac{s_{k^*} + \triangle s_{k^*}}{w_{k^*}}} - \frac{1}{\frac{1}{a_j w_j} + \frac{s_j + \triangle s_j}{w_j}} > 0 \tag{38}$$

and $\underline{\sigma}_i = \overline{\sigma}_i = 0$, as $\chi_i \subset E$. If the set χ_i implies $\sum_{j\in\chi_i} \triangle s_j = 0$, then let

$$\underline{\sigma}_i = \lambda - \frac{1}{\frac{1}{a_{k_i^*(\chi_i)} w_{k^*(\chi_i)}} + \frac{s_{k^*(\chi_i)}}{w_{k^*(\chi_i)}}} \ge 0, \tag{39}$$

and $\overline{\sigma}_i = 0$. Also, if $s_j > 0$, let $\mu_j = 0$; if $s_j = 0$, let $\mu_j = \lambda - \underline{\sigma}_i - a_j w_j \ge 0$, keeping the mentioned values of $\underline{\sigma}_i$ and $\overline{\sigma}_i$.

On the other hand, if $i \in \Lambda$ and $j \in \chi_i$, then

$$0 \quad < \quad \sum_{j\in\chi_i} \triangle s_j = \overline{P}_i - \underline{P}_i, \tag{40}$$

$$\overline{\sigma}_i \quad = \quad \frac{1}{\frac{1}{a_{k^*(\chi_i)} w_{k^*(\chi_i)}} + \frac{s_{k^*(\chi_i)} + \triangle s_{k^*(\chi_i)}}{w_{k^*(\chi_i)}}} - \lambda \ge 0, \tag{41}$$

let $\underline{\sigma}_i = 0$ and $\mu_j = 0$, as $\triangle s_j > 0$. If $\triangle s_j = 0$, then

$$\mu_j = \lambda + \overline{\sigma}_i - \frac{1}{\frac{1}{a_j w_j} + \frac{s_j}{w_j}} > 0 \tag{42}$$

and $\underline{\sigma}_i = 0$.

Therefore, there have been the Lagrange multipliers $\lambda, \{\underline{\sigma}_i, \overline{\sigma}_i\}_{i=1}^T$ and $\{\mu_k\}_{k=1}^K$ obtained above, the Lagrange function of which, for the problem (30), is:

$$\begin{aligned} & L(\{s_k\}, \lambda, \{\overline{\sigma}_i\}, \{\underline{\sigma}_i\}, \{\mu_k\}) \\ = \; & \textstyle\sum_{k=1}^K w_k \log(1 + a_k s_k) - \lambda \left(\sum_{k=1}^K s_k - P\right) \\ - \; & \textstyle\sum_{i=1}^T \overline{\sigma}_i (\sum_{j\in\chi_i} s_j - \overline{P}_i) + \sum_{i=1}^T \underline{\sigma}_i (\sum_{j\in\chi_i} s_j - \underline{P}_i) \\ + \; & \textstyle\sum_{k=1}^K \mu_k s_k. \end{aligned} \tag{43}$$

Further, by observation, they satisfy the KKT conditions. Since the problem (30) is a differentiable convex optimization problem with linear constraints, not only are the KKT conditions mentioned above sufficient, but they are also necessary for optimality. We can observe that the constraint qualification of the problem (30) holds. **Proposition 3.2** is hence proved.

Remark 3.3. If we chose the CWF to solve the problem (30), similarly, a non-linear system with non-linear equations and inequalities in multiple dual variables would have had to

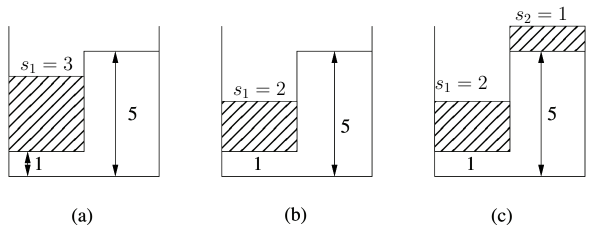

Fig. 2. Illustration for Example 1, results for CWF and GWFPP. (a) CWF: without peak power restriction check ($s_1 = 3, s_2 = 0$). (b) CWF: s_1 is clipped considering peak power constraint. (c) GWFPP: $s_1 = 2, s_2 = 1$.

be solved:

$$\begin{cases} \sum_{i=1}^{T} \sum_{j\in\chi_i} \left(\frac{w_j}{\lambda+\overline{\sigma}_i-\underline{\sigma}_i} - \frac{1}{a_i}\right)^+ = P; \\ \underline{P}_i \le \sum_{j\in\chi_i} \left(\frac{w_j}{\lambda+\overline{\sigma}_i-\underline{\sigma}_i} - \frac{1}{a_i}\right)^+ \le \overline{P}_i, \\ \text{as } i = 1,2,\ldots,T; \\ \underline{\sigma}_i (\sum_{j\in\chi_i} \left(\frac{w_j}{\lambda+\overline{\sigma}_i-\underline{\sigma}_i} - \frac{1}{a_i}\right)^+ - \underline{P}_i) = 0, \\ \text{as } i = 1,2,\ldots,T; \\ \overline{\sigma}_i (\sum_{j\in\chi_i} \left(\frac{w_j}{\lambda+\overline{\sigma}_i-\underline{\sigma}_i} - \frac{1}{a_i}\right)^+ - \overline{P}_i) = 0, \\ \text{as } i = 1,2,\ldots,T; \\ \lambda \ge 0; \underline{\sigma}_i \ge 0, \overline{\sigma}_i \ge 0, \text{ as } i = 1,2,\ldots,T. \end{cases} \tag{44}$$

There seems no existing result that can solve this system.

IV. Numerical Results and Complexity Analysis

As an illustration for the proposed algorithm, some numerical examples are provided in this section.

Example 1. Instance a case of the water-filling with individual peak power constraints (WFPP) problem:

$$\begin{array}{ll} \max_{\{s_i\}_{i=1}^2} & \sum_{i=1}^2 \log(1 + a_i s_i) \\ \text{subject to:} & 0 \le s_i \le 2, \ \forall i; \\ & \sum_{i=1}^2 s_i \le 3, \end{array} \tag{45}$$

where $a_1 = 1$ and $a_2 = 0.2$. The problem given is a WFPP problem. In Fig.2, the step depth for channel 1 and channel 2 are 1 and 5 respectively, as the reciprocal of their respective channel gains. Using the CWF, the solution is shown in Fig. 2(a): all the power is allocated to the first channel with good channel condition. If consider peak power constraints check, s_1 may be clipped as shown in Fig. 2(b).

Utilizing the proposed Algorithm **GWFPP**, the result of the first loop is $s_1 = 2$, as part of the solution based on the algorithm. The remaining of the solution, s_2, is allocated with zero. The result of the second loop is $s_1 = 2$ and $s_2 = 1$, as full entries of the solution. According to **Proposition 3.1**, the result of the second loop is guaranteed to be the optimal solution. With the proposed **GWFPP**, we can obtain the optimal solution as shown in Fig. 2(c).

Example 2. Instance another case of the water-filling with the WFPP problem with multiple channels:

$$\begin{array}{ll} \max_{\{s_i\}_{i=1}^8} & \sum_{i=1}^8 \log(1 + a_i s_i) \\ \text{subject to:} & 0 \le s_i \le i, \ \forall i; \\ & \sum_{i=1}^8 s_i \le 30, \end{array} \tag{46}$$

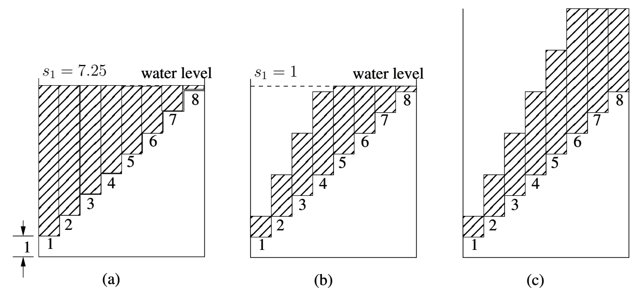

Fig. 3. Illustration for power allocation using CWF and the proposed GWFPP for Example 2. (a) Results for CWF. (b) CWF, clipped s_1 to s_4 due to peak power restrictions. (c) Results for GWFPP.

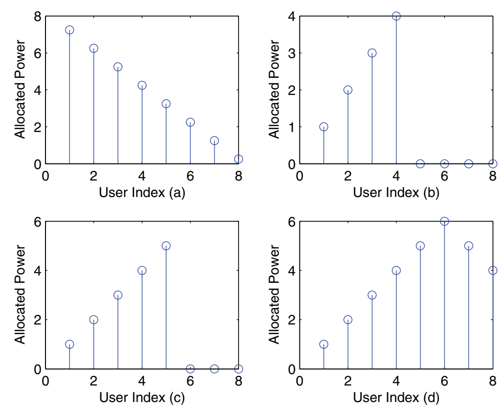

Fig. 4. Optimal power allocation results for Example 2. (a) solution of CWF; (b) 1st iteration results of GWFPP; (c) 2nd iteration results of GWFPP; (d) 3rd iteration results of GWFPP.

where $a_i = 1/i, \forall i$. The step depth is then monotonically increase from 1 to 8, as shown in Fig. 3. For the CWF, without considering the peak power constraints, the water level is solved as 8.25, and then the power allocation is shown in Fig. 3(a) and Fig. 4(a). Considering peak power constraints, the power levels for channels 1-4 are clipped and are set to their peak values as shown in Fig. 3(b). The CWF doesn't tell us where to and how to assign the clipped power.

Utilizing the proposed Algorithm **GWFPP**, the result of the first loop is $s_i = i$, as $i = 1, \ldots, 4$, as part of the solution. The remaining entries of the solution are allocated with zero, as shown in Fig. 4(b). The result of the second loop is $s_i = i$, as $i = 1, \ldots, 5$, also as part of the solution. The remaining entries of the solution are allocated with zero, as shown in Fig. 4(c). The results of the third loop are $s_i = i$, as $i = 1, \ldots, 5$; and $s_i = 12 - i$, as $i = 6, 7, 8$, as full entries of the solution, as shown in Fig. 4(d) and Fig. 3(c). According to **Proposition 3.1**, the result of the third loop is the optimal solution.

It is shown that the solution for the problem with sum power constraint only (see for example, Fig. 3(a) and Fig. 4(a)) is different from the solution of the corresponding problem with added peak power constraints (see for example, Fig. 3(c) and Fig. 4(d)). From Fig. 3, we can observe that for more complicated problems, the conventional water-fill exhibits its limitations. The water level is no longer a unique level. Thus, our approach using the concept of water-fill is more general to solve the RRA problems.

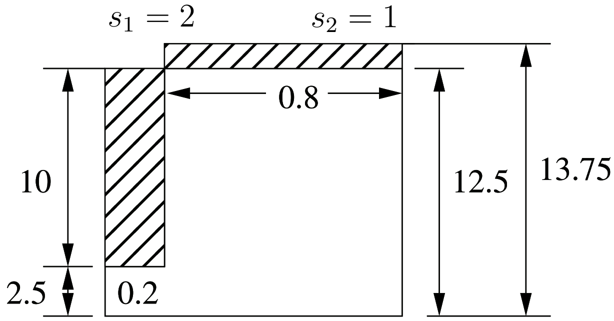

Fig. 5. Illustration for Example 3 using GWFPP ($s_1 = 2, s_2 = 1$).

Example 3. Instance a case of the weighted water-filling with individual peak power constraints (WFPP) problem:

$$\begin{array}{ll} \max_{\{s_i\}_{i=1}^2} & \sum_{i=1}^2 w_i \log(1 + a_i s_i) \\ \text{subject to:} & 0 \le s_i \le 2, \ \forall i; \\ & \sum_{i=1}^2 s_i \le 3, \end{array} \tag{47}$$

where $a_1 = 2, a_2 = 0.1, w_1 = 0.2$ and $w_2 = 0.8$. Utilizing the proposed Algorithm **GWFPP**, the result of the first loop is $s_1 = 2$, as part of the solution. The remaining of the solution, s_2, is allocated with zero. The result of the second loop is $s_1 = 2$ and $s_2 = 1$, as full entries of the solution. From **Proposition 3.1**, the result of the second loop is guaranteed to be the optimal solution. The result is illustrated in Fig. 5. In this figure, for channel 1, the stair width is 0.2, specified by its weight factor. The level depth is $1/(a_1 w_1) = 2.5$. Similarly, for the channel two, the stair width is 0.8 and the level depth is $1/(a_1 w_1) = 12.5$. The power allocated for channel 1 is 2, so the water level for channel 1 is 12.5. For channel 2, the power is 1, the water level is 13.75. Again, water level is not unique for different channels.

Example 4. As the last example, we instance a case of the weighted water-filling with group bounded power constraints (WFGBP) problem:

$$\begin{array}{ll} \max_{\{s_i\}_{i=1}^3} & \sum_{i=1}^3 w_i \log(1 + a_i s_i) \\ \text{subject to:} & 0 \le s_i, \ \forall i; \\ & \sum_{i=1}^3 s_i \le 5; \\ & 1 \le s_1 + s_2 \le 2.5; \\ & 1 \le s_3 \le 2.5, \end{array} \tag{48}$$

where $a_1 = a_2 = a_3 = 1, w_1 = 0.3, w_2 = 0.2$ and $w_3 = 0.5$. Utilizing the proposed Algorithm **GWFGBP**, the result from 1)-3) of the **GWFGBP** is: $s_1 = 0.8, s_2 = 0.2$ and $s_3 = 1$. Then continuously using 4)-6) of the **GWFGBP**, the optimal solution is: $s_1{}^* = 0.8 + 0.9 = 1.7, s_2{}^* = 0.2 + 0.6 = 0.8$ and $s_3{}^* = 1 + 1.5 = 2.5$.

The results are shown in Fig.6, where the stair width for the three channels are 0.3, 0.2, 0.5 respectively specified by their weighting factors. The step depth is calculated as $1/(a_i w_i)$, leading to the step depth values as 3.33, 5, and 2 respectively for the three channels. The water level for channel 1 and channel 2 is the same, but different with that of channel 3. It is interesting to observe that with the same path gain of these three channels, the channel with the highest weight factor, channel 3, has even lower water level due to the structure of the constraints.

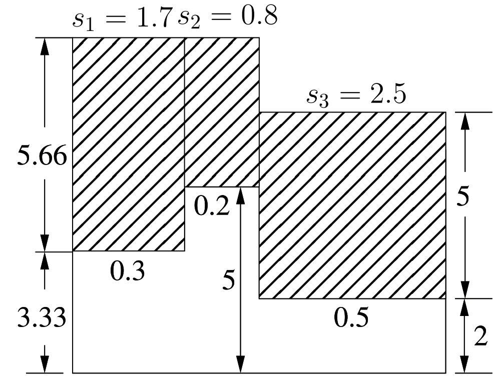

Fig. 6. Illustration for Multi-User Power Allocation Results by the GWFGBP.

A. Complexity Analysis

As stated in [16] (Section 3), the conventional WF algorithm had an exponential worst-case complexity of 2^K, where K is the number of the channels, even though the channel gains had been sorted in decreasing order. Pointing to this case, [16] proposed an improved algorithm with worst-case complexity of K iterations. Since each iteration consists of multiple arithmetic and logical operations, here we use total number of operations as a measure of the complexity level (See [23], Chapter 8).

The CWF approach has a worst-case complexity of K iterations, *i.e.*, total $O(K^2)$ fundamental arithmetic and logical operations under the $2(K + 1)$ memory requirement and the sorted parameters $\{w_k a_k\}_{k=1}^K$ (e.g. see [24], pp 137, for more details).

The proposed **GWF** algorithm occupies less computational resource. It is seen that it needs K loops at most to search k^* and it needs 4 arithmetic operations and 2 logical operations to complete each loop. Thus, the worst-case computational complexity of the proposed solution is $8K + 3$ (from the operations of $6K + 3 + 2K$) fundamental arithmetical and logical operations under the $2(K + 1)$ memory units to store $\{d_i\}, \ \{w_i\}, \ W_s$, and P_M.

For the **GWFPP**, it needs K loops to compute the optimal solution, at most. The required number of operations is, at worst, $\sum_{i=1}^K (8i + 3) = 4K^2 + 7K$ fundamental arithmetical and logical operations.

For the **GWFGBP**, it needs T loops to compute the optimal solution, at most. The required number of operations, at worst, is $O(K^2)$ fundamental arithmetical and logical operations.

In this complexity analysis, we didn't take sorting procedure into consideration. It is stated in [16] that the channel gain sequences come from the eigenvalues of a matrix and many of the algorithms to compute the eigenvalues and eigenvectors already produce the eigenvalues sorted.

V. Conclusion

In this paper, we proposed a new geometric approach for the well-known unweighted and weighted water-filling problems to solve a class of radio resource allocation (RRA) problems. The proposed **GWF** approach makes use of the geometric relation among the channel gains, the allocated power and the total power. It provides straightforward power allocation

TABLE I
LIST OF VARIABLES AND ABBREVIATIONS

variables & abbreviations	representations or interpretations
a_i	propagation path gain for the ith channel
d_i	step depth $(= 1/a_i)$ of the ith stair
k^*	water level step (highest step under water)
E	a subsequence of the sequence $\{1, 2, \cdots K\}$
$\|E\|$	cardinality of the set E
K	total number of channels
P	total power
$P_2(k)$	power (water volume) above step k
$\underline{P}_i$	lower bound for power level for the ith group
$\overline{P}_i$	upper bound for power level for the ith group
s_i	power allocated for the ith channel
s_{k^*}	power allocated to the water level step
w_i	weight for the ith channel
$\delta_{i,j}$	step depth difference $(= d_i - d_j)$
λ	reciprocal of water level μ
μ	water level
CWF	conventional water-filling
GWF	geometry water-filling
WFPP	water-filling with individual peak power constraints
WFGBP	water-filling with group bounded power constraints
RRA	Radio resource allocation

analysis, solutions and insights with reduced computation over conventional approach.

The **GWF** is further extended to more general forms: the **GWFPP** and then the **GWFGBP** to solve the weighted water-filling RRA problems. The WFPP refers the problem with individual peak power constraints; while the WFGBP refers the problem with group bounded power constraints. This kind of problems is more general to model a communication system with different constraints.

Our proposed **GWFPP** and **GWFGBP** algorithms compute the optimal solutions to the WFPP and the WFGBP with moderate complexity. Its optimality has been proven in this paper. Numerical examples are provided to illustrate the effectiveness of the proposed algorithms. Our results also show that with the complicated problem structure, the conventional water-filling approach is limited due to the fact that the water levels are no longer unique.

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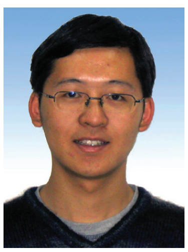

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